PMM U.S.S.R.,Vol.49,No.5,pp.641-647,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

STUDY OF THE AXISYMMETRIC CONTACT PROBLEM OF THE WEAR OF A PAIR CONSISTING OF AN ANNULAR STAMP AND A ROUGH HALF-SPACE*

E.V. KOVALENKO

The solution of on axisymmetric contact problem of wear in an elastic, rough half-space by an annular stamp, is used to construct a method of investigating a class of two-dimensional integral equations of the second kind containing Fredholm coordinate operators and Volterra time operators. By applying to these equations an analogue of the method of separation of variables, we can reduce the problem to consecutive solutions of integral Fredholm and Volterra equations of the second kind, and in the case of the Fredholm equation the problem solved is that of determining the eigenvalues and eigenfunctions.

The method enables us to obtain expansions for the basic characteristics of the phenomenon of contact interaction, valid over the whole range of time changes. In the case of contact problems with wear and smooth bodies, the basic solution equations can be studied using the method of matched asymptotic expansions. The integral equation of the second kind with a logarithmic kernel obtained in the boundary layer, is solved in a semi-infinite interval in closed form by reducing it to a functional difference equation with shear.

1. As we know /1, 2/, the axisymmetric contact problem with abrasive wear, of impressing a stamp annular in the plane $(c \leqslant r \leqslant 1)$ into a rough, elastic half-space, written in dimensionless coordinates using the notation given in /1/, can be reduced to an integral equation (here and henceforth we assume that the quantity T is sufficiently large, but such that $\gamma(t)$ is of the order of the displalements in the linear theory of elasticity)

$$l\varphi(r,t) + \frac{2}{\pi}\int_c^1 \varphi(\rho,t) K\left(\frac{2\sqrt{\rho r}}{\rho+r}\right)\frac{\rho\, d\rho}{\rho+r} = \tag{1.1}$$
$$\gamma(t) - f(r) - r\int_c^t \varphi(r,\tau)\, d\tau \quad (0 < c \leqslant r \leqslant 1,\ 0 \leqslant t \leqslant T < \infty)$$

($K(k)$ is the complete elliptic integral of the first kind), when the following condition is satisfied:

$$P(t) = 2\pi \int_c^1 r\varphi(r,t)\, dr \tag{1.2}$$

Clearly /1/, two basic versions of problem (1.1), (1.2) are of interest: 1) the function $\gamma(t)$, characterizing the displacement of the stamp, regarded as a rigid unit, with time, is given, while the contact pressure $\varphi(r,t)$ and the force $P(t)$, pressing the stamp to the foundation are determined; 2) $P(t)$ is given, and we determine $\varphi(r,t)$ and $\gamma(t)$. In both cases formulas (1.2), (1.3) of /1/ are then used to find the foundation wear rate.

Let us consider the first version of system (1.1), (1.2). Let the rigid displacement of the stamp $\gamma(t)$ vary with time according to the law

$$\gamma(t) = \gamma + \gamma_\infty t + \gamma_*(t) \quad (0 \leqslant t \leqslant T) \tag{1.3}$$
$$\gamma_*(t) \to 0 \ (t \to \infty);\ \gamma,\ \gamma_\infty = \mathrm{const}$$

Henceforth, we shall assume that the function $\gamma(t)$ is continuous for $t \in [0, T]$, i.e. $\gamma(t) \in C(0, T)$. Then, as was shown in /1/, the contact stresses $\varphi(r,t)$ have the following structure:

$$\varphi(r,t) = r^{-1}\varphi_\infty + \varphi_*(r,t),\ \varphi_\infty = \gamma_\infty \tag{1.4}$$
$$\varphi_*(r,t) \to 0 \ (t \to \infty)$$

Using expressions (1.4), we shall simplify integral Eq.(1.1) as follows:

*Prikl.Matem.Mekhan.,49,5,836-843,1985

$$lr\varphi(r,t)+H\rho\varphi(\rho,t)=\gamma(t)-f(r)-r\int_0^t\varphi(r,\tau)\,d\tau-l\Phi(r) \tag{1.5}$$

$$\Phi(r)=\frac{1-r}{T}\int_0^T\varphi(r,t)\,dt$$

$$(c\leqslant r\leqslant 1,\quad 0\leqslant t\leqslant T)$$

$$H\varphi=\frac{2}{\pi}\int_c^1\varphi(\rho)K\left(\frac{2\sqrt{\rho r}}{\rho+r}\right)\frac{d\rho}{\rho+r} \tag{1.6}$$

and subtract from it, term by term, the relation obtained from (1.5) at $t=0$. Taking into account formulas (1.3), (1.4), we obtain

$$lr[\varphi_*(r,t)-\varphi_*(r,0)]+H\rho[\varphi_*(\rho,t)-\varphi_*(\rho,0)]+ \tag{1.7}$$
$$r\int_0^t\varphi_*(r,\tau)\,d\tau=\gamma_*(t)-\gamma_*(0)$$
$$(c\leqslant r\leqslant 1,\ 0\leqslant t\leqslant T)$$

Let us transfer from Eq.(1.6) to the system of two integral equations equivalent to it /3, 4/

$$l[\varphi_1(r,t)-\varphi_1(r,0)]+H[\varphi_1(\rho,t)-\varphi_1(\rho,0)]+\int_0^t\varphi_1(r,\tau)\,d\tau=0 \tag{1.8}$$

$$l[\varphi_2(r,t)-\varphi_2(r,0)]+H[\varphi_2(\rho,t)-\varphi_2(\rho,0)]+\int_0^t\varphi_2(r,\tau)\,d\tau=\gamma_*(t)-\gamma_*(0) \tag{1.9}$$

$$(c\leqslant r\leqslant 1,\ 0\leqslant t\leqslant T)$$
$$\varphi_*(r,t)=r^{-1}[\varphi_1(r,t)+\varphi_2(r,t)] \tag{1.10}$$

It follows that we require to find a general solution of the homogeneous Eq.(1.8) and a solution of the inhomogeneous Eq.(1.9).

Let us construct the solution of (1.8). We first note that the operator H (1.6) is self-conjugate, completely continuous and positive definite /1/; it acts from the Hilbert space $L_2(c,1)$ into $L_2(c,1)$. According to the general theory of such operators /5/, a system of eigenfunctions $\{\varphi_n(r)\}$ $(n\geqslant 1)$ $H\varphi$ (1.6) is orthogonal and complete in $L_2(c,1)$, all eigenvalues λ_n of the operator $H\varphi$ are real and positive, and $\lambda_1>\lambda_2>\ldots>\lambda_n>\ldots>0$, $\lambda_n\sim n^{-1}$ $(n\to\infty)$. The algorithm for constructing the eigenfunctions of the operator H (1.6) is given in /1, 4/.

We shall seek the solution of (1.8) in the form

$$\varphi_1(r,t)=\sum_{n=1}^{\infty}a_n^{(1)}(t)\varphi_n(r) \tag{1.11}$$

Substituting (1.11) into (1.8) and taking into account the relation

$$H\varphi_n=\lambda_n\varphi_n(r)\quad(c\leqslant r\leqslant 1;\ n=1,2,\ldots) \tag{1.12}$$

we obtain

$$\sum_{n=1}^{\infty}(l+\lambda_n)\varphi_n(r)[a_n^{(1)}(t)-a_n^{(1)}(0)]+\sum_{n=1}^{\infty}\varphi_n(r)\int_0^t a_n^{(1)}(\tau)\,d\tau=0$$

Equating in the last expression the coefficients on the left and right-hand sides of eigenfunctions of the operator H with the same index, we arrive at the following equations for determining the function $a_n^{(1)}(t)$ $(n\geqslant 1)$:

$$(l+\lambda_n)[a_n^{(1)}(t)-a_n^{(1)}(0)]+\int_0^t a_n^{(1)}(\tau)\,d\tau=0 \tag{1.13}$$
$$(0\leqslant t\leqslant T)$$

and this yields

$$a_n^{(1)}(t)=a_n^{(1)}(0)\exp(-\alpha_n t),\quad \alpha_n=(l+\lambda_n)^{-1};\quad \alpha_n\to l^{-1} \tag{1.14}$$

Let us now construct the solution of the inhomogeneous Eq.(1.9), seeking it in the form

$$\varphi_2(r,t)=\sum_{n=1}^{\infty}a_n^{(2)}(t)\varphi_n(r) \tag{1.15}$$

We substitute (1.12), (1.15) into (1.9) and use the fact the eigenfunctions of the operator H are orthonormalized. As before, we obtain

$$(l+\lambda_n)[a_n^{(2)}(t)-a_n^{(2)}(0)]+\int_0^t a_n^{(2)}(\tau)\,d\tau = b_n[\gamma_*(t)-\gamma_*(0)] \quad (n=1,2,\ldots) \tag{1.16}$$

$$\left(1=\sum_{n=1}^{\infty} b_n\varphi_n(r), \quad b_n=\int_c^1 \varphi_n(r)\,dr\right)$$

Now taking into account the asymptotic formulas (1.14), we expand the functions $\gamma_*(t)$ and $a_n^{(2)}(t)$ in exponential series /6/ in $\{\exp(-\alpha_n t)\}$, converging uniformly for $t\in[0,T]$

$$\gamma_*(t)=\sum_{n=1}^{\infty}\delta_n\exp(-\alpha_n t) \tag{1.17}$$

$$a_n^{(2)}(t)=\sum_{i=1}^{\infty}{}' c_{ni}\exp(-\alpha_i t)+c_{nn}t\exp(-\alpha_n t)$$

(the prime on the summation sign in (1.17) means that the term with index $n=i$ is omitted). Substituting (1.17) into (1.16) and equating the coefficients of $(\exp(-\alpha_n t)-1)$ we obtain

$$c_{ni}=\delta_i b_n\alpha_n\alpha_i(\alpha_i-\alpha_n)^{-1} \quad (i\neq n), \quad c_{nn}=-\delta_n b_n\alpha_n^2 \tag{1.18}$$

Thus we construct the solution of the problem in question in the form (1.3)–(1.5), (1.11), (1.14), (1.15) and (1.17), to within a denumerable set of constants $a_n^{(1)}(0)$ $(n\geqslant 1)$. To find the unknown constants we write, in accordance with the above formulas,

$$\varphi(r,0)=r^{-1}\left[\varphi_\infty+\sum_{n=1}^{\infty}a_n^{(1)}(0)\varphi_n(r)+\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}{}' c_{ni}\varphi_n(r)\right] \tag{1.19}$$

$$\Phi(r)=(1-r)r^{-1}\Big\{\varphi_\infty-T^{-1}\sum_{n=1}^{\infty}\alpha_n^{-1}a_n^{(1)}(0)\times \tag{1.20}$$

$$(\exp(-\alpha_n T)-1)\varphi_n(r)-T^{-1}\sum_{n=1}^{\infty}\Big[\sum_{i=1}^{\infty}{}'\alpha_i^{-1}c_{ni}(\exp(-\alpha_i T)-1)+$$

$$\alpha_n^{-1}c_{nn}T\exp(-\alpha_n T)+\alpha_n^{-2}(\exp(-\alpha_n T)-1)c_{nn}\Big]\varphi_n(r)\Big\}$$

and use integral Eq.(1.5) for $t=0$, where the function $f(r)\in L_2(c,1)$ has been previously expanded in a series in eigenfunctions of the operator H (1.6)

$$f(r)=\sum_{n=1}^{\infty}f_n\varphi_n(r), \quad f_n=\int_c^1 f(r)\varphi_n(r)\,dr \tag{1.21}$$

converging in the norm of the space $L_2(c,1)$. We also note that according to calculation all terms within the curly brackets of (1.20) except the first, can be neglected when $T\geqslant 0$ and we can still achieve an accuracy that is sufficient in practice. After this, substituting (1.19)–(1.21) into (1.5) for $t=0$, multiplying both sides of the latter by $\varphi_m(r)$, integrating from c to 1 and using the condition of orthonormalization of the eigenfunctions of (1.6), we obtain

$$a_m^{(1)}(0)=\alpha_m\Big\{b_m\Big(\gamma+\sum_{n=1}^{\infty}\delta_n\Big)-[\gamma_\infty(lb_m+d_m+le_m)+f_m]\Big\}-\sum_{n=1}^{\infty}{}' c_{mn} \tag{1.22}$$

$$d_m=(1-c)\int_c^1\varphi_m(r)B(r)\,dr, \quad e_m=\int_c^1\frac{1-r}{r}\varphi_m(r)\,dr$$

$$B(r)=\frac{2}{\pi(1-c)}\int_c^1 K\left(\frac{2\sqrt{yr}}{y+r}\right)\frac{dy}{y+r}$$

Finally, the general solution of the problem in question will have the form

$$\varphi(r,t)=r^{-1}\Big[\gamma_\infty+\sum_{n=1}^{\infty}a_n^{(1)}(0)\varphi_n(r)\times \tag{1.23}$$

$$\exp(-\alpha_n t)+\sum_{m=1}^{\infty}\psi_m(t)\varphi_m(r)\Big]$$

$$\psi_m(t)=\sum_{n=1}^{\infty}{}' c_{mn}\exp(-\alpha_n t)+tc_{mn}\exp(-\alpha_m t)$$

and the constants $a_n^{(1)}(0)$ are given in the form (1.22).

Now using the condition of statics (1.2) we obtain, in accordance with formula (1.23),

$$P(t)=2\pi\Big[(1-c)\gamma_\infty+\sum_{n=1}^{\infty}a_n^{(1)}(0)b_n\times\exp(-\alpha_n t)+\sum_{m=1}^{\infty}b_m\psi_m(t)\Big] \tag{1.24}$$

i.e. the force pressing the stamp into the foundation and the contact pressure both tend to a constant value with time.

Following the procedure in /4/, we can show that the series (1.23) converges in $L_2(c, 1)$ uniformly with respect to t in the interval $[0, T]$ $(T \geqslant 0)$, and Eq.(1.23) gives the generalized solution of Eq.(1.1).

2. Now let

$$P(t) = P_\infty + P_*(t);\ P_\infty = \text{const},\ P_*(t) \to 0\ (t \to \infty) \tag{2.1}$$

Then, as was shown in Sect.1, the contact stress $\varphi(r, t)$ and the subsidence of the foundation surface under the stamp $\gamma(t)$ changes with time according to the laws (1.3), (1.4), and $P_\infty = 2\pi(1-c)\varphi_\infty = 2\pi(1-c)\gamma_\infty$ provided that we demand, in accordance with the condition (1.2) and representation

$$\varphi_*(r, t) = r^{-1}[\varphi_1^*(t) + \varphi_2(r, t)] \tag{2.2}$$

that

$$\int_c^1 \varphi_2(r, t)\,dr = 0, \quad 2\pi\int_c^1 \varphi_1^*(t)\,dr = 2\pi(1-c)\varphi_1^*(t) = P_*(t) \tag{2.3}$$

We now change, as before, from the integral Eq.(1.1) to the approximate Eq.(1.7). If we now add to, and subtract from, the left-hand side of the latter the expressions

$$Dr[\varphi_1^*(t) - \varphi_1^*(0)], \quad \int_c^1 [\varphi_2(\rho, t) - \varphi_2(\rho, 0)]\rho B(\rho)\,d\rho$$

then the integral Eq.(1.7) will be satisfied when the functions $\varphi_1^*(t)$ and $\varphi_2(r, t)$ supply solutions to the corresponding equations

$$(l + D)[\varphi_1^*(t) - \varphi_1^*(0)] + \int_0^t \varphi_1^*(\tau)\,d\tau = \tag{2.4}$$

$$\gamma_*(t) - \gamma_*(0) - \int_c^1 [\varphi_2(\rho, t) - \varphi_2(\rho, 0)]B(\rho)\,d\rho$$

$$l[\varphi_2(r, t) - \varphi_2(r, 0)] + \frac{2}{\pi}\int_c^1 [\varphi_2(\rho, t) - \varphi_2(\rho, 0)]k(\rho, r)\,d\rho + \tag{2.5}$$

$$\int_0^t \varphi_2(r, \tau)\,d\tau = g(r, t)$$

$$(c \leqslant r \leqslant 1,\ 0 \leqslant t \leqslant T)$$

$$k(\rho, r) = \frac{1}{\rho + r} K\left(\frac{2\sqrt{\rho r}}{\rho + r}\right) - \pi/2[B(\rho) + B(r)] \tag{2.6}$$

$$g(r, t) = [\varphi_1^*(t) - \varphi_1^*(0)][D - (1-c)B(r)]$$

Note that the kernel $k(\rho, r)$ of the form (2.6) is symmetric and has the property

$$\int_c^1\int_c^1 \varphi_2(\rho, t)k(\rho, r)\,d\rho\,dr = 0 \tag{2.7}$$

Let us introduce the space $L_2^0(c, 1)$ of functions square integrable in the segment $[c, 1]$ satisfying the condition

$$\int_c^1 g(r)\,dr = 0 \tag{2.8}$$

We can show that the space $L_2^0(c, 1)$ is a subspace of $L_2(c, 1)$.

Theorem. The integral operator

$$H^0\varphi = \frac{2}{\pi}\int_c^1 \varphi(\rho)k(\rho, r)\,d\rho \tag{2.9}$$

is a selfconjugate, completely continuous and positive operator acting in $L_2^0(c, 1)$.

The first two assertions of the theorem follow directly from the symmetry of the kernel (2.6), its quadratic summability and condition (2.7).

Let us establish the positiveness of the operator (2.9) in $L_2^0(c, 1)$. To do this, we shall use the relation /7/

$$\frac{2}{\pi(r + \rho)} K\left(\frac{2\sqrt{r\rho}}{\rho + r}\right) = \int_0^\infty J_0(ru)J_0(\rho u)\,du$$

Next, constructing the scalar product for $\varphi \not\equiv 0$, we obtain

$$(\varphi, H^0\varphi)_{L_2^0} = \int_0^\infty F^2(u)\,du > 0, \quad F(u) = \int_c^1 \varphi(r)J_0(ur)\,dr$$

the latter implying the positiveness of the operator H^0 in $L_2^0(c, 1)$.

We will use the methods of /1, 4/ to construct a system of eigenfunctions $\{\varphi_n(r)\}$ and the corresponding sequence of eigenvalues $\{\lambda_n\}$ of the operator (2.9).

According to the theorem the system is orthonormal and complete in $L_2^0(c, 1)$, and all $\lambda_n \geqslant 0$ with $\lambda_n \sim n^{-1}$ $(n \to \infty)$. The function $\varphi_2(r, t)$, appearing in Eq.(2.5) will be sought in the form (2.10)

$$\varphi_2(r, t) = \varphi_2^{(1)}(r, t) + \varphi_2^{(2)}(r, t) \tag{2.10}$$

where $\varphi_2^{(1)}(r, t)$ is the general solution of the homogeneous Eq.(2.5) and $\varphi_2^{(2)}(r, t)$ is the solution of the same inhomogeneous equation. Choosing the constant D in (2.6) in such a manner that $g(r, t) \in L_2^0(c, 1)$ for any t (see (2.8)), we shall write the functions $\varphi_2^{(1)}(r, t), \varphi_2^{(2)}(r, t)$ and $g(r, t)$ in the form of the following series:

$$\varphi_2^{(i)}(r, t) = \sum_{n=1}^{\infty} a_n^{(i)}(t)\varphi_n(r) \quad (i = 1, 2) \tag{2.11}$$

$$g(r, t) = [\varphi_1^*(t) - \varphi_1^*(0)] \sum_{n=1}^{\infty} b_n\varphi_n(r) \tag{2.12}$$

$$b_n = \frac{\pi(1-c)}{2} \int_c^1 \varphi_n(r) B(r)\, dr$$

The homogeneous Eq.(2.5) and the first relation of (2.11) are both reduced to the integral Eq.(1.13) whose solution has the form (1.14). Substituting the second equation of (2.11) and (2.12) into the Eq.(2.5) and equating the coefficients on the left and right sides of the resulting expression for the eigenfunctions of the operator H^0 with like index, we obtain

$$\alpha_n^{-1}[a_n^{(2)}(t) - a_n^{(2)}(0)] + \int_0^t a_n^{(2)}(\tau)\, d\tau = b_n[\varphi_1^*(t) - \varphi_1^*(0)] \quad (0 \leqslant t \leqslant T) \tag{2.13}$$

Since the system of exponents $\{\exp(-\alpha_n t)\}$, taking the last condition of (1.14) into account, is closed in $C(0, T)$ /6/, paying due regard to the formula (2.3), we obtain,

$$\frac{1}{2\pi(1-c)} P_*(t) = \varphi_1^*(t) = \sum_{n=1}^{\infty} \delta_n \exp(-\alpha_n t) \tag{2.14}$$

Eq.(2.13), according to (2.14), will be satisfied if $a_n^{(2)}(t)$ has the structure (1.17), (1.18).

Finally, using the relations (2.10), (2.11), (2.14), (1.14) and (1.17) we obtain from (2.4) the additional subsidence, unknown under the stamp, and the functions $\varphi_2(r, t), \Phi(r)$

$$\gamma_*(t) = \sum_{n=1}^{\infty} \{[(D + \lambda_n)\delta_n + a_n^{(1)}(0)\beta_n]\exp(-\alpha_n t) + \beta_n\psi_n(t)\}, \tag{2.15}$$

$$\beta_n = \int_c^1 \varphi_n(r) B(r)\, dr$$

$$\varphi_2(r, t) = \sum_{n=1}^{\infty} [a_n^{(1)}(0)\exp(-\alpha_n t) + \psi_n(t)]\varphi_n(r) \tag{2.16}$$

$$\Phi(r) = (1 - r) r^{-1}\varphi_\infty \tag{2.17}$$

Here, as before, when $T \geqslant 0$ the remaining terms in (2.17) for $\Phi(r)$ can be neglected compared with those retained.

The unknown constants $a_n^{(1)}(0)$ in Eq.(2.16) are found by substituting the contact stress $\varphi(r, 0)$ into integral Eq.(1.5) and $t = 0$, and carrying out the following transformations. We supplement the system $\{\varphi_n(r)\}$ $(n \geqslant 1)$ of eigenfunctions of the operator H^0 (2.9) with the element $\varphi_0(r) = (1 - c)^{-1/2}$. Then the sequence of functions $\{\varphi_n(r)\}$ $(n \geqslant 0)$ will be orthonormalized and complete in the space $L_2(c, 1)$. Let us expand the functions $f(r)$, $(1 - c)B(r)$, $(1 - r)r^{-1} \in L_2(c, 1)$ in series in terms of the system $\{\varphi_n(r)\}$ $(n \geqslant 0)$

$$f(r) = \sum_{n=0}^{\infty} f_n\varphi_n(r), \quad (1 - c)B(r) = \sum_{n=0}^{\infty} d_n\varphi_n(r) \tag{2.18}$$

$$(1 - r)r^{-1} = \sum_{n=0}^{\infty} e_n\varphi_n(r) \quad (\{f_n\}, \{d_n\}, \{e_n\} \in l_2)$$

converging, at least, over the norm of the space $L_2(c, 1)$. From formulas (2.1)–(2.3), (2.17) we find

$$\varphi(r, 0) = \frac{1}{r}\sum_{n=0}^{\infty} X_n\varphi_n(r), \quad X_0 = \frac{1}{2\pi\sqrt{1-c}} P(0) \tag{2.19}$$

$$X_n = a_n^{(1)}(0) + \sum_{m=1}^{\infty}{}' c_{nm} \quad (n \geqslant 1)$$

Now using Eqs. (2.18), (2.19) and taking into account the orthonormality of the system of functions $\{\varphi_n(r)\}$ $(n \geqslant 0)$ we obtain, in accordance with (1.5) for $t = 0$

$$X_n = -\alpha_n (X_0 d_n (1-c)^{-1/2} + f_n + le_n) \quad (n \geqslant 1) \tag{2.20}$$
$$X_0 = -\frac{1}{l\sqrt{1-c} + d_0}\Big[\sum_{n=1}^{\infty} X_n d_n + \sqrt{1-c}\,(f_0 + le_0) - (1-c)\,\gamma(0)\Big]$$

When writing Eqs. (2.20), we have taken into consideration the fact that $\varphi_n(r)$ $(n \geqslant 1)$ are the eigenfunctions of the operator (2.9) and satisfy condition (2.8). From (2.19), (2.20), we find the constants $a_n^{(1)}(0)$, then determine $\gamma_*(t)$ using Eq. (2.15), and use the last relation of (2.20) to obtain the constant γ in (1.3).

An assertion analogous to that formulated at the end of Sect.1 can be proved for the function $\varphi(r, t)$.

3. Let us consider some special cases. Let e.g. in (1.3) $\gamma_\infty = \gamma_*(t) \equiv 0$. This means that the progressive displacement of the stamp is independent of time /2/. From Eqs. (1.4), (1.22)–(1.24) we find that

$$a_m^{(1)}(0) = \alpha_m (\gamma b_m - f_m - le_m) \quad (m \geqslant 1) \tag{3.1}$$
$$\varphi(r, t) = r^{-1} \sum_{n=1}^{\infty} a_n^{(1)}(0)\, \varphi_n(r) \exp(-\alpha_n t), \qquad P(t) = 2\pi \sum_{n=1}^{\infty} a_n^{(1)}(0)\, b_n \exp(-\alpha_n t)$$

i.e. in this case the constant pressure and the force pressing the stamp into the foundation both tend to zero with time.

Let us now assume that in (1.3) only $\gamma_*(t) \equiv 0$. This means that the rate of progressive displacement of the stamp is constant in time and equal to γ_∞. From (1.17), (1.18) we find that $\delta_n = c_{ni} = 0$.

Let us also study the limit version of the integral Eq. (1.1) when $l = 0$. Making the change of variable $r = c\exp[(1+x)\lambda]$, $\rho = c\exp[(1+\xi)/\lambda]$ and introducing the notation given by the formulas $\psi(\xi, t) = \rho^{3/2}\varphi(\rho, t)$, $\lambda = -2(\ln c)^{-1}$, $\beta(x, t) = \lambda\sqrt{r}\,[\gamma(t) - f(r)]$, $m(z) = \operatorname{sch}(^1/_2 z)\, K[\operatorname{sch}(^1/_2 z)]$, we obtain an integral equation of the first kind with a difference kernel

$$\frac{1}{\pi}\int_{-1}^{1} \psi(\xi, t)\, m\left(\frac{\xi - x}{\lambda}\right) d\xi = \beta(x, t) - \lambda \int_0^t \psi(x, \tau)\, d\tau \quad (|x| \leqslant 1) \tag{3.2}$$
$$m(z) \sim -\ln|z| \quad (z \to 0) \tag{3.3}$$

Let us further introduce the small parameter $\varepsilon\lambda^{-1}$ $(\varepsilon\lambda^{-1} \ll 1)$ and rewrite (3.2) in the form

$$\frac{1}{\pi}\int_{-1}^{1} \psi(\xi, \varepsilon)\, m\left(\frac{\xi - x}{\lambda}\right) d\xi = \beta(x, \varepsilon) - \lambda \int_0^{\varepsilon\lambda^{-1}} \psi(x, \tau)\, d\tau \quad (|x| \leqslant 1) \tag{3.4}$$

$$\frac{1}{\pi}\int_{-1}^{1} [\psi(\xi, t) - \psi(\xi, \varepsilon)]\, m\left(\frac{\xi - x}{\lambda}\right) d\xi = \beta(x, t) - \beta(x, \varepsilon) - \tag{3.5}$$
$$\lambda \int_{\varepsilon\lambda^{-1}}^{t} \psi(x, \tau)\, d\tau \quad (|x| \leqslant 1,\ \varepsilon\lambda^{-1} \leqslant t \leqslant T < \infty)$$

Note that the solution of the integral Eq. (3.5) can be obtained using the methods of Sects.1 and 2, and we shall therefore not describe its construction. Let us consider Eq. (3.4). Taking into account the fact that $\psi(x, t) = \psi(x, \varepsilon) + O(\varepsilon\lambda^{-1})$ $(t \in (0, \varepsilon\lambda^{-1}),\ \varepsilon\lambda^{-1} \ll 1)$, and neglecting in the resulting expression terms of the order of $(\varepsilon\lambda^{-1})^2$, we obtain

$$\varepsilon\psi(x, \varepsilon) + \frac{1}{\pi}\int_{-1}^{1} \psi(\xi, \varepsilon)\, m\left(\frac{\xi - x}{\lambda}\right) d\xi = \beta(x, \varepsilon) \quad (|x| \leqslant 1) \tag{3.6}$$

The solution of (3.6) was obtained using the method of matched asymptotic expansions and has the form /8/

$$\psi(x, \varepsilon) = \frac{1}{\sqrt{1 - x^2}}\left[\omega(x) - \frac{\omega(1)}{2}\left(\sqrt{1+x} + \sqrt{1-x}\right)\right] + \frac{\omega(1)}{\sqrt{2\varepsilon}}\left[q\left(\frac{1+x}{\varepsilon}\right) + q\left(\frac{1-x}{\varepsilon}\right)\right] \tag{3.7}$$

We have in (3.7), by virtue of (3.3) $\omega(x) = \psi(x, 0)\sqrt{1 - x^2} \in C(-1, 1)$, and the function $\psi(x, 0)$ satisfies integral Eq. (3.6) for $\varepsilon = 0$, and $q(s)$ is a solution of the equation

$$q(s) - \frac{1}{\pi}\int_0^{\infty} q(\tau) \ln\left|\frac{\tau - s}{\tau}\right| d\tau = q(0) \quad (0 \leqslant s < \infty)$$

As was shown in /8/, the latter expression is equivalent to the functional difference equation with shear /9/, which has a closed solution.

The authro thanks N.Kh. Arutyunyan and V.M. Aleksandrov for their interest.

REFERENCES

1. ALEKSANDROV V.M. and KOVALENKO E.V., An axisymmetric contact problem for a linearly deformable foundation of general type, with wear. Izv. Akad. Nauk SSSR, MTT, 5, 1978.
2. GALIN L.A. and GORYACHEVA I.G., An axisymmetric contact problem of the theory of elasticity, with wear. PMM 41, 5, 1977.
3. ALEKSANDROV V.M. and KOVALENKO E.V., On a class of integral equations of mixed problems of the mechanics of continuous media. Dokl. Akad. Nauk SSSR, 252, 2, 1980.
4. KOVALENKO E.V., On the approximate solution of a certain type of integral equation of the theory of elasticity and mathematical physics. Izv. Akad. Nauk ArmSSR, Mekhanika, 34, 5, 1981.
5. KANTOROVICH L.V. and AKILOV G.P., Functional Analysis, Moscow, Nauka, 1977.
6. LEONT'EV A.F., Exponential Series. Moscow, Nauka, 1976.
7. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products. Moscow, Fizmatgiz, 1963.
8. ALEKSANDROV V.M. and KOVALENKO E.V., On contact interaction of bodies with coverings when there is wear. Dokl. Akad. Nauk SSSR, 275, 4, 1984.
9. MIROLYUBOV A.A. and SOLDATOV M.A., Linear, Homogeneous Difference Equations. Moscow, Nauka, 1981.

Translated by L.K.

PMM U.S.S.R.,Vol.49,No.5,pp.647-650,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

HIGH-FREQUENCY SHEAR OSCILLATIONS OF A STRIP STAMP ON AN ELASTIC HALF-SPACE*

A.S. ZIL'BERGLEIT, I.N. ZLATINA and T.YU. SIMKINA

Harmonic shear oscillations of a rigid stamp with a plane base coupled to an elastic half-space are studied. The problem is reduced to dual integral equations whose solution, effective for short waves, was constructed in /1/ and validated in /2/. Expressions for the complex amplitude of stamp oscillations are given and analysed, as well as those for the contact stresses and their intensity coefficients, and the power averaged over the period of oscillations transmitted from the stamp to the half-space per unit length of the stamp.

The problems of the oscillation of a strip stamp on an elastic half-space have been the subject of a considerable number of publications (see /3, 4/ and their bibliography). However, a high-frequency analysis of the oscillation patterns is practically non-existent. The only known reference on the subject in a short section in /5/.

Consider a rigid strip stamp whose plane foot occupies a strip $|x| < a, y = 0, |z| < \infty$. The stamp is coupled to a homogeneous isotropic elastic half-space $y > 0$ and executes oscillations along the Oz axis under the action of a load harmonically time-dependent, with linear density $\operatorname{Re}(Te^{i\omega t})$. We will write the unique non-zero displacement component in the form

$$u_z = u_z(x, y, t) = \operatorname{Re}[w(x, y)\, e^{i\omega t}]$$

$$\Delta w + k^2 w = 0,\ y > 0;\ k = \omega c^{-1},\ c = G^{1/2}\rho^{-1/2} \tag{1}$$

$$w|_{y=0} = \delta, \quad |x| < a; \quad \left.\frac{\partial w}{\partial y}\right|_{y=0} = 0, \quad |x| > a \tag{2}$$

The function $w(x, y)$ satisfies the demand of local finiteness of the energy and radiation conditions; G is the shear modulus, δ is the unknown complex amplitude of the stamp oscillations and c is the velocity of volume shear waves. The problem is closed by the equation of motion of the stamp in complex form

**Prikl.Matem.Mekhan.*,49,5,844-848,1985